CHAPTER 1
SEQUENCE AND SERIES

A collection or set of numbers, arranged in order, according to some pattern or law, so that one number can be identified as the first and another as the second, and so forth, is referred to as a sequence. The word sequence is sometimes replaced by the word progression, but we will use sequence. The set of natural numbers forms a sequence; that is, 1, 2, 3, ... is a sequence. Each number in the sequence is called a term, and we will represent the first term of a sequence by the letter (a) and the last term by the letter (ℓ).

ARITHMETIC SEQUENCES

An arithmetic sequence is a sequence in which each term may be determined from the preceding term by the addition of a constant. This constant, called the common difference, is designated by the letter (d) and will maintain the same value throughout the sequence.

An arithmetic sequence, then, may be indicated by a, a + d, a + 2d, a + 3d, ..., a + (n - 1)d, where there are n terms in the sequence. In the sequence

$$-1, 3, 7, \ldots$$

the common difference (d) is 4. The first term (a) is -1; therefore, the second term is

$$a + d = -1 + 4 = 3$$

The third term is

$$a + 2d = -1 + 2(4) = 7$$

and the fourth term is

$$a + 3d = -1 + 3(4) = 11$$

EXAMPLE: Find the next three terms in the sequence 5, 9, 13, ...

SOLUTION: The first term (a) is 5 and the difference (d) is 4; therefore, write

$$a = 5$$

$$d = 4$$

then, the fourth term is

$$a + (n - 1)d = 5 + (4 - 1)\,4$$

$$= 17$$

the fifth term is

$$a + (n - 1)d = 5 + (5 - 1)\,4$$

$$= 21$$

and the sixth term is

$$5 + (6 - 1)\,4 = 25$$

We are often interested in finding a specific term of a sequence. We usually refer to this term as the nth term. In cases where the nth term is the last term of a sequence, we write

$$\ell = a + (n - 1)d$$

In this formula there are four unknowns. If we know any three of them, we may find the fourth.

EXAMPLE: Find the 20th term of the sequence

$$1, 3, 5, 7, \ldots .$$

SOLUTION: We know that

$$a = 1$$

$$d = 2$$

$$n = 20$$

Therefore,

$$\ell = a + (n - 1)d$$

$$= 1 + (20 - 1)2$$

$$= 39$$

Notice that we let the nth (20th) term be the last term.

EXAMPLE: Find the number of terms in a sequence if the last term is 37, the difference is 5, and the first term is -13.

SOLUTION: We know that

$$a = -13$$

$$d = 5$$

$$\ell = 37$$

We solve

$$\ell = a + (n - 1)d$$

for

$$n$$

by writing

$$\ell = a + (n - 1)d$$

$$\ell - a = (n - 1)d$$

$$\frac{\ell - a}{d} = n - 1$$

$$\frac{\ell - a}{d} + 1 = n$$

By substitution

$$n = \frac{\ell - a}{d} + 1$$

$$= \frac{37 - (-13)}{5} + 1$$

$$= \frac{50}{5} + 1$$

$$= 10 + 1$$

$$= 11$$

There are cases where we desire the first term of a sequence when only two terms are known.

EXAMPLE: What is the first term of the sequence if the third term is 8 and the sixth term is 20?

SOLUTION: We write

$$\_\ \_\ \underset{8}{\_}\ \_\ \_\ \underset{20}{\_}$$

and consider a sequence where

$$a = 8$$

$$\ell = 20$$

and

$$n = 4$$

Therefore,

$$20 = 8 + (4 - 1)d$$

then

$$d = \frac{20 - 8}{3}$$

$$= 4$$

Now consider a sequence where

$$\ell = 20$$

$$d = 4$$

and

$$n = 6$$

then write

$$20 = a + (6 - 1)4$$

$$20 - 20 = a$$

$$a = 0$$

PROBLEMS: Write the next two terms in the following sequences.

1. 18, 21, 24, . . .

2. -19, -16, -13, . . .

3. x, x + 2, x + 4, . . .

4. $\sqrt{2}$ + 3, $\sqrt{2}$ + 7, $\sqrt{2}$ + 11, . . .

Find the term asked for in the following sequences.

5. Seventh term of $-\frac{1}{2}$, 0, $\frac{1}{2}$, $\cdots$

6. Twenty-fifth term of -19, -10, -1, ...

7. Fifth term of $-6x^2$, $-2x^2$, $2x^2$, ...

8. Which term of -2, 3, 8, ... is 88?

ANSWERS:

1. 27, 30

2. -10, -7

3. x + 6, x + 8

4. $\sqrt{2} + 15$, $\sqrt{2} + 19$

5. $\frac{5}{2}$

6. 197

7. $10x^2$

8. 19th

ARITHMETIC MEANS

In the sequence 1, 3, 5, 7, the terms 3 and 5 occur between the first term 1 and the last term 7 and are designated the means. Generally, the terms which occur between two given terms are called the means.

If we are given the first term (a) and the last term (ℓ) in a sequence of n terms, then there are (n - 2) means between a and ℓ. There can be any number of means between two given terms of a sequence, depending on the difference between adjacent terms.

To determine the means of a sequence we use the formula

$$\ell = a + (n - 1)d$$

EXAMPLE: Insert two arithmetic means between 6 and 12.
SOLUTION: We know

$$a = 6$$

$$\ell = 12$$

and that there are two means in this sequence; therefore,

$$n = 4$$

because the means plus two (the first and last terms) is the number of terms in the sequence. We now determine the difference by writing

$$\ell = a + (n - 1)d$$

$$\ell - a = (n - 1)d$$

$$\frac{\ell - a}{n - 1} = d$$

and by substitution

$$d = \frac{12 - 6}{4 - 1}$$

$$d = \frac{6}{3}$$

$$d = 2$$

We now add this difference to the first term to obtain the second term and the difference to the second term to obtain the third term as follows:

$$6 + 2 = 8 = \text{second term}$$

$$8 + 2 = 10 = \text{third term}$$

and find the means are 8 and 10.

We could also have used the general form of a sequence (a, a + d, a + 2d, a + 3d, ...) and added the difference to the first term to obtain the second term and added two times the difference to the first term to obtain the third term.

If we use the same first and last terms, that is,

$$a = 6$$

and

$$\ell = 12$$

but now ask for six means, we still use the same formula

$$\ell = a + (n - 1)d$$

and in this case the number of terms is the six means plus the first and last term or

$$n = 8$$

Therefore,

$$d = \frac{\ell - a}{n - 1}$$

$$= \frac{12 - 6}{8 - 1}$$

$$= \frac{6}{7}$$

and the means are

$$a + d = 6 + \frac{6}{7}$$

$$a + 2d = 6 + \frac{12}{7}$$

$$a + 3d = 6 + \frac{18}{7}$$

$$a + 4d = 6 + \frac{24}{7}$$

$$a + 5d = 6 + \frac{30}{7}$$

$$a + 6d = 6 + \frac{36}{7}$$

The previous examples demonstrate that there can be any number of means between two given terms of a sequence, depending on the number of terms and the difference.

PROBLEMS: Insert the indicated number of means in the following:

1. Three, between 3 and 19

2. Two, between -10 and -4

3. Five, between -2 and 2

4. Two, between the first and fourth terms if the fourth term is six and the seventh term is eleven.

5. A secretary can type 3 words per minute faster for each half-hour she types. If she starts at 8:30 a.m. at the rate of 35 words per minute, how fast is she typing at 10:00 a.m.?

ANSWERS:

1. 7, 11, 15

2. -8, -6

3. $-\frac{4}{3}$, $-\frac{2}{3}$, 0, $\frac{2}{3}$, $\frac{4}{3}$

4. $\frac{8}{3}$, $\frac{13}{3}$

5. 44 words per minute

ARITHMETIC SERIES

When we add all the terms of a sequence, we call this indicated sum a series. We will use the symbol S_n to designate the indicated sum of n terms of a sequence. To derive a formula for S_n we may write the terms of a series as

$$S_n = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d]$$

Notice that when we had the second term we added the difference to obtain the third term. If we had the third term and desired the second term we would have to subtract the difference. Therefore, if we write the series with the last term first, we subtract the difference for each succeeding term. Then, using ℓ to represent the last term, write

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + [\ell - (n - 1)d]$$

Now add the two equations, term by term, and find

$$S_n = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d]$$

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + [\ell - (n - 1)d]$$

and

$$2S_n = (a + \ell) + (a + \ell) + (a + \ell) + \cdots + (a + \ell)$$

where there are n times $(a + \ell)$. Therefore,

$$2S_n = n(a + \ell)$$

$$S_n = \frac{n}{2}(a + \ell)$$

Another way of obtaining the formula for S_n is to write the terms of a series as follows:

$$S_n = a + (a + d) + (a + 2d) + \cdots$$

$$+ (\ell - 2d) + (\ell - d) + \ell$$

then reverse the order of the series and combine both equations as follows:

$$S_n = a + (a + d) + (a + 2d) + \cdots$$

$$+ (\ell - 2d) + (\ell - d) + \ell$$

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots$$

$$+ (a + 2d) + (a + d) + a$$

and find that

$$2S_n = (a + \ell) + (a + \ell) + \cdots$$

$$+ (a + \ell) + (a + \ell) + (a + \ell)$$

$(a + \ell)$ occurs n times which yields

$$2S_n = n(a + \ell)$$

$$S_n = \frac{n}{2}(a + \ell)$$

EXAMPLE: Find the sum of the first 5 terms of the series 2, 6, 10, ...
SOLUTION: We know that

$$a = 2$$

$$d = 4$$

and

$$n = 5$$

and write

$$S_n = \frac{n}{2}(a + \ell)$$

Here, we must determine ℓ by

$$\ell = a + (n - 1)d$$

$$= 2 + (5 - 1)\,4$$

$$= 2 + 16$$

$$= 18$$

Then,

$$S_n = \frac{n}{2}(a + \ell)$$

$$= \frac{5}{2}(2 + 18)$$

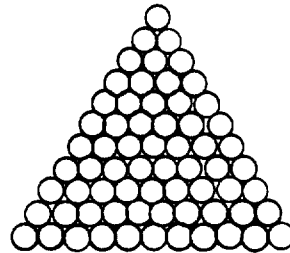$$S_n = \frac{5}{2}(20)$$

$$= 50$$

This may be verified by writing

$$2, 6, 10, 14, 18$$

and adding the terms to find

$$S_n = 50$$

EXAMPLE: There are eleven pieces of pipe in the bottom row of a stack of pipe which form a triangle. How many pieces of pipe are in the stack?
SOLUTION: We know pipe is stacked as shown.

We have in our stack eleven pieces of pipe on the bottom row and each row up contains one less piece of pipe. There is one piece of pipe on the top.
Therefore, we write

$$a = 1$$

$$n = 11$$

$$d = 1$$

$$\ell = 11$$

Then

$$S_n = \frac{n}{2}(a + \ell)$$

$$= \frac{11}{2}(1 + 11)$$

$$= \frac{11}{2}(12)$$

$$= 66 \text{ pieces of pipe in the stack}$$

EXAMPLE: Find the sum of 23 terms of the series -3, 2, 7, ...

SOLUTION: We know that

$$a = -3$$

$$d = 5$$

and

$$n = 23$$

Write

$$S_n = \frac{n}{2} (a + \ell)$$

We do not know ℓ, but we do know that

$$\ell = a + (n - 1)d$$

and by substituting for ℓ in the equation for the sum, we have

$$S_n = \frac{n}{2} (a + \ell)$$

$$= \frac{n}{2} [a + a + (n - 1)d]$$

$$= \frac{n}{2} [2a + (n - 1)d]$$

Then, using the values we know

$$S_n = \frac{23}{2} [2(-3) + (23 - 1)5]$$

$$= \frac{23}{2} [-6 + (22)5]$$

$$= \frac{23}{2} [-6 + 110]$$

$$= \frac{23}{2} [104]$$

$$= 23 [52]$$

$$= 1,196$$

In some cases we may have to work from the sum of a sequence in order to determine the sequence.

EXAMPLE: Find the first 4 terms of the sequence if

$$a = 2$$

$$\ell = 18$$

$$S_n = 200$$

SOLUTION: We must find d. We write

$$\ell = a + (n - 1)d$$

but notice that there are two unknowns; that is, n and d. We then write

$$S_n = \frac{n}{2} (a + \ell)$$

and by substitution

$$200 = \frac{n}{2} (2 + 18)$$

$$400 = n (20)$$

$$20 = n$$

We again write

$$\ell = a + (n - 1)d$$

and substitute, then write

$$18 = 2 + (20 - 1)d$$

$$\frac{18 - 2}{19} = d$$

$$d = \frac{16}{19}$$

We know that

$$a = 2$$

and

$$d = \frac{16}{19}$$

Therefore, the first 4 terms are

$$a$$

$$a + d$$

$$a + 2d$$

$$a + 3d$$

which give

$$2$$

$$2 + \frac{16}{19}$$

$$2 + \frac{32}{19}$$

$$2 + \frac{48}{19}$$

and the first 4 terms are

$$2, \; 2\frac{16}{19}, \; 3\frac{13}{19}, \; 4\frac{10}{19}$$

PROBLEMS: Find the sum of the sequence having

1. $a = 3$, $\ell = 7$, $n = 15$

2. $a = -6$, $\ell = 18$, $n = 6$

3. $a = 7$, $n = 5$, $d = 2$

4. $d = 6$, $\ell = 32$, $n = 5$

5. $a = -19$, $d = 3$, $\ell = -7$

In problems 6 and 7, find the first 4 terms if

6. $a = 6$, $\ell = 14$, $S_n = 50$

7. $n = 7$, $\ell = 20$, $S_n = 70$

ANSWERS:

1. 75

2. 36

3. 55

4. 100

5. -65

6. 6, 8, 10, 12

7. $0, \; 3\frac{1}{3}, \; 6\frac{2}{3}, \; 10$

GEOMETRIC SEQUENCES

A geometric sequence (or progression) is a sequence (or progression) in which each term is a multiple of any other, with a constant ratio between adjacent terms. This constant, called the common ratio, is designated by the letter r and will maintain the same value throughout the sequence.

A geometric sequence, then, may be indicated by a, ar, ar^2, ... $ar^{(n-1)}$, where there are n terms in the sequence. The common ratio (r) in a geometric sequence may be determined by dividing any term by its preceding term. The quotient is the common ratio.

In the sequence

$$2, 6, 18, \ldots$$

the common ratio r is 3. The first term is a and is equal to 2. This may be shown by

$$a = 2$$

$$ar = 2 \cdot 3$$

$$ar^2 = 2 \cdot 3^2$$

If there are n terms in the sequence, then the last term is

$$ar^{(n-1)}$$

Notice that if we considered this sequence to have only three terms then the last term would be

$$\ell = ar^{(n-1)}$$

$$= 2 \cdot 3^2$$

$$= 18$$

EXAMPLE: Find the next three terms in the sequence

$$3, 12, 48, \ldots$$

SOLUTION: The first term a is 3 and the ratio is

$$r = \frac{12}{3}$$

$$= 4$$

Therefore, the fourth, fifth, and sixth terms are

$$ar^{(4-1)}, ar^{(5-1)}, \text{ and } ar^{(6-1)}$$

Then,

$$a(r)^{(4-1)} = 3(4)^3$$
$$= 3 \cdot 64$$
$$= 192$$

and

$$a(r)^{5-1} = 3(4)^4$$
$$= 3 \cdot 256$$
$$= 768$$

and

$$a(r)^{6-1} = 3(4)^5$$
$$= 3 \cdot 1024$$
$$= 3072$$

EXAMPLE: Find the last term of the sequence where

$$a = 3$$
$$n = 5$$

and

$$r = 2$$

SOLUTION: Write

$$\ell = a(r)^{(n-1)}$$
$$= 3(2)^{5-1}$$
$$= 3(2)^4$$
$$= 48$$

EXAMPLE: Find the first term of a sequence if the second term is 6, the third term is 24, and the fourth term is 96.

SOLUTION: We consider the last term as 96 and write

$$\ell = a(r)^{(n-1)}$$

We desire a, but we do not know r. To find r we divide any term by the preceding term; that is,

$$\frac{96}{24} = 4$$

or

$$\frac{24}{6} = 4$$

and find r to be 4.

Substitution yields

$$96 = a(4)^{(4-1)}$$

where n is 4 because 96 is the fourth term. Then,

$$96 = a(4)^3$$
$$= a(64)$$

and

$$a = \frac{96}{64}$$
$$= \frac{3}{2}$$

The sequence is

$$\frac{3}{2}, 6, 24, 96.$$

PROBLEMS: Write the first three terms of each sequence if

1. $a = 2, r = 5$

2. $a = -3, r = \frac{1}{2}$

3. $a = 1, r = .01$

Find the last term of each sequence

4. $a = 7, n = 5, r = 2$

5. $a = \frac{1}{2}, n = 4, r = \frac{1}{3}$

6. $30, 10, 3\frac{1}{3}, \ldots$ and $n = 6$

ANSWERS:

1. 2, 10, 50

2. -3, $-\frac{3}{2}$, $-\frac{3}{4}$

3. 1, .01, .0001

4. 112

5. $\frac{1}{54}$

6. $\frac{10}{81}$

GEOMETRIC MEANS

In the sequence 5, 15, 45, 135, the terms 15 and 45 occur between the first term 5 and the last term 135 and are designated the means. Generally, the terms which occur between two given terms are called the means.

If we are given the first term (a) and the last term (ℓ) in a sequence of n terms, then there are (n - 2) means between a and ℓ. There can be any number of means between two given terms of a sequence, depending on the common ratio between adjacent terms.

In order that the means between terms in a sequence may be inserted, the common ratio must be known.

To find the means between two terms of a sequence we use the formula

$$\ell = a(r)^{n-1}$$

EXAMPLE: Insert two means between 3 and 24.

SOLUTION: We consider 3 the first term and 24 the last term and write

$$\underline{3} \ \_ \ \_ \ \underline{24}$$

There are four terms in the sequence and we write

$$\ell = a(r)^{n-1}$$

By substitution

$$24 = 3(r)^3$$

$$\frac{24}{3} = r^3$$

$$r^3 = 8$$

$$r = 2$$

Now find the means underlined

$$a, \ \underline{ar}, \ \underline{ar^2}, \ ar^3$$

to be

$$ar = 3 \cdot 2$$

$$= 6$$

and

$$ar^2 = 3(2)^2$$

$$= 3 \cdot 4$$

$$= 12$$

EXAMPLE: Insert a mean between $\frac{1}{2}$ and $\frac{9}{32}$.

SOLUTION: We consider there are three terms and write

$$a = \frac{1}{2}$$

$$\ell = \frac{9}{32}$$

$$n = 3$$

therefore,

$$\ell = a(r)^{n-1}$$

$$\frac{9}{32} = \frac{1}{2}(r)^2$$

$$r^2 = \frac{9}{32} \div \frac{1}{2}$$

$$r^2 = \frac{9}{16}$$

$$r = \pm\sqrt{\frac{9}{16}}$$

$$r = \frac{3}{4} \text{ or } -\frac{3}{4}$$

We find the mean we desire is the second term and write

$$ar = \frac{1}{2} \cdot \frac{3}{4}$$

$$= \frac{3}{8}$$

and the sequence is

$$\frac{1}{2}, \frac{3}{8}, \frac{9}{32}$$

or

$$ar = \frac{1}{2} \left(-\frac{3}{4}\right)$$

$$= -\frac{3}{8}$$

and the sequence is

$$\frac{1}{2}, -\frac{3}{8}, \frac{9}{32}$$

In cases where only one mean is required, the following may be used. We know the common ratio may be found by dividing any term by the preceding term; that is, if the sequence is a, m, ℓ, then the common ratio is either

$$\frac{m}{a} = r$$

or

$$\frac{\ell}{m} = r$$

therefore,

$$\frac{m}{a} = \frac{\ell}{m}$$

and

$$m^2 = a\ell$$

$$m = \sqrt{a\ell} \text{ or } -\sqrt{a\ell}$$

In the previous example where we wanted to find the one mean between $\frac{1}{2}$ and $\frac{9}{32}$ we could have written

$$m = \pm \sqrt{a\ell}$$

$$= \pm \sqrt{\frac{1}{2} \cdot \frac{9}{32}}$$

$$= \pm \sqrt{\frac{9}{64}}$$

$$= \frac{3}{8} \text{ or } -\frac{3}{8}$$

PROBLEMS: Insert the indicated number of means in the sequences.

1. Two, between 3 and 24

2. Two, between $\frac{1}{2}$ and $\frac{1}{54}$

3. One, between 4 and 9

4. Three, between x^2 and x^{10}

5. Four, between -5 and $-\frac{5}{243}$

ANSWERS:

1. 6, 12

2. $\frac{1}{6}, \frac{1}{18}$

3. 6 or -6

4. x^4, x^6, x^8 or $-x^4, x^6, -x^8$

5. $-\frac{5}{3}, -\frac{5}{9}, -\frac{5}{27}, -\frac{5}{81}$

GEOMETRIC SERIES

When we add all the terms of a sequence, we call this indicated sum a series. We use the symbol S_n to designate the indicated sum of n terms of a sequence. To derive a formula for S_n we may write the terms of a series as

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

then multiply each term of the series by (-r) to obtain

$$-rS_n = -ar - ar^2 - ar^3 - \cdots - ar^{n-1} - ar^n$$

and combine the two equations as follows:

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$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$

$$-rS_n = \quad - ar - ar^2 - \cdots - ar^{n-2} - ar^{n-1} - ar^n$$

$$S_n - rS_n = a \qquad\qquad\qquad - ar^n$$

$$S_n(1 - r) = a - ar^n$$

$$S_n = \frac{a - ar^n}{1 - r}$$

$$= \frac{a(1 - r^n)}{1 - r}, \; r \neq 1$$

EXAMPLE: Find the sum of six terms in the series whose first term is 3 and whose common ratio is 2.

SOLUTION: We know

$$a = 3$$

$$r = 2$$

and

$$n = 6$$

therefore,

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{3(1 - 2^6)}{1 - 2}$$

$$= \frac{3(1 - 64)}{-1}$$

$$= \frac{3(-63)}{-1}$$

$$= \frac{-189}{-1}$$

$$= 189$$

In cases where we know the last term, the first term, and the common ratio and desire S_n we could use

$$S_n = \frac{a - ar^n}{1 - r}$$

but we would first have to determine n.

EXAMPLE: Find the sum of a series if

$$a = 3, \; r = 4, \; \text{and} \; \ell = 192.$$

SOLUTION: To find n we write

$$\ell = ar^{n-1}$$

and by substitution

$$192 = (3)(4)^{n-1}$$

$$\frac{192}{3} = 4^{n-1}$$

$$64 = 4^{n-1}$$

then

$$64 = 4^3$$

and

$$4^3 = 4^{n-1}$$

Therefore,

$$3 = n - 1$$

$$n = 4$$

Now

$$S_n = \frac{a - ar^n}{1 - r}$$

$$= \frac{3 - 3(4)^4}{1 - 4}$$

$$= \frac{3 - 768}{-3}$$

$$= \frac{-765}{-3}$$

$$= 255$$

In order to decrease the number of operations in the previous example we may write

$$S_n = \frac{a - ar^n}{1 - r}$$

and

$$ar^n = r(ar^{n-1})$$

Therefore,

$$S_n = \frac{a - r(ar^{n-1})}{1 - r}$$

but

$$ar^{n-1} = \ell$$

then

$$S_n = \frac{a - r\ell}{1 - r}$$

The solution to the previous problem would be

$$S_n = \frac{3 - 4(192)}{1 - 4}$$

$$= \frac{3 - 768}{-3}$$

$$= \frac{-765}{-3}$$

$$= 255$$

PROBLEMS: Find S_n in the following series if

1. $a = 3$, $r = 5$, and $n = 4$

2. $a = \frac{1}{2}$, $r = \frac{1}{3}$, and $n = 3$

3. $a = -\frac{1}{3}$, $r = 6$, and $n = 4$

4. $a = 4$, $r = 3$, and $\ell = 324$

5. $a = \frac{2}{3}$, $r = -\frac{1}{2}$, and $\ell = -\frac{1}{12}$

6. $a = \frac{5}{3}$, $r = 3$, and $\ell = 32,805$

ANSWERS:

1. 468

2. $\frac{13}{18}$

3. $-86\frac{1}{3}$

4. 484

5. $\frac{5}{12}$

6. $49,206\frac{2}{3}$

INFINITE SERIES

As previously discussed, a series is the indicated sum of the terms of a sequence. If the number of terms of a series is unlimited, the series is said to be infinite; that is,

$$1 + 2 + 4 + \cdots$$

is an infinite series and

$$1 + 2 + 4 + \cdots + n$$

is a finite series because there is a finite number of terms.

When we desire the sum of a geometric series such as

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

and we know the number of terms, we use the formula

$$S_n = \frac{a - ar^n}{1 - r}$$

This formula may be written as

$$S_n = \frac{a - ar^n}{1 - r}$$

$$= \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

If we increase the number of terms desired, notice that the second term

$$\frac{ar^n}{1 - r}$$

becomes larger if $|r| > 1$ and becomes smaller if $|r| < 1$.

When the number of terms of a series continues indefinitely, the series is an infinite series. Therefore, in

$$\frac{ar^n}{1 - r}$$

as $n \to \infty$ the term goes to ∞ if $|r| > 1$ and the term goes to zero if $|r| < 1$. When the term

$$\frac{ar^n}{1 - r}$$

goes to ∞, the sum of the series is not defined. However, if this term goes to zero, we may write the sum of the series as

$$\lim_{n \to \infty} S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

$$= \frac{a}{1 - r} - 0$$

$$= \frac{a}{1 - r}$$

which is the sum of an infinite series when $|r| < 1$. We designate the limit of the sum of an infinite series as

$$S = \frac{a}{1 - r}$$

EXAMPLE: Find the sum of the infinite series

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots$$

SOLUTION: Determine that

$$r = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

Then

$$S = \frac{a}{1 - r}$$

$$= \frac{\frac{1}{3}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{2}}$$

$$= \frac{2}{3}$$

which is the limiting value of the infinite series.

EXAMPLE: Find the sum of the infinite series

$$7 + \frac{7}{5} + \frac{7}{25} + \cdots$$

SOLUTION: Determine that

$$r = \frac{\frac{7}{5}}{7}$$

$$= \frac{1}{5}$$

then

$$S = \frac{a}{1 - r}$$

$$= \frac{7}{1 - \frac{1}{5}}$$

$$= \frac{7}{\frac{4}{5}}$$

$$= \frac{35}{4}$$

$$= 8\frac{3}{4}$$

PROBLEMS: Find S in the following:

1. $1 + \frac{2}{3} + \frac{4}{9} + \cdots$

2. $6 + 2 + \frac{2}{3} + \cdots$

3. $.1 + .01 + .001 + \cdots$

4. $\sqrt{2} + 1 + \frac{\sqrt{2}}{2} + \cdots$

ANSWERS:

1. 3

2. 9

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3. $\dfrac{1}{9}$

4. $2\sqrt{2} + 2$

THE nth TERM

In cases where we are given the nth term of a series, it is relatively easy to find other terms from the symbolic definition of the nth term.

EXAMPLE: If the nth term of a series is given by

$$\frac{n}{2n + 1}$$

find the first three terms.

SOLUTION: To determine the first term replace n by 1 and for the second term replace n by 2, etc.; that is, the first term is

$$\frac{n}{2n + 1} = \frac{1}{2(1) + 1}$$

$$= \frac{1}{3}$$

and the second term is

$$\frac{2}{2(2) + 1}$$

$$= \frac{2}{5}$$

and the third term is

$$\frac{3}{2(3) + 1}$$

$$= \frac{3}{7}$$

To perform the converse of this type problem, that is, to find the nth term of a given series, is quite different because there are no set rules which may be applied. Also, there may be many formulas which express the nth term of a series.

EXAMPLE: Find the nth term of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

SOLUTION: The numerator of the terms remains the same and is 1. The denominator follows a regular pattern of increasing by 2 for each term.

If we write

term	1	2	3	4
numerator	1	1	1	1
denominator	2	4	6	8

we see that each term's denominator may be written as

$$2 \cdot n$$

Therefore, the nth term for the series is

$$\frac{1}{2n}$$

and the series may be designated as

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n} + \cdots$$

PROBLEMS: Find the first 3 terms of the series whose n^{th} term is given by

1. n^2

2. $\dfrac{1}{n^2 + 3}$

3. $\dfrac{n^2}{n + 1}$

ANSWERS:

1. $1 + 4 + 9 + \cdots$

2. $\dfrac{1}{4} + \dfrac{1}{7} + \dfrac{1}{12} + \cdots$

3. $\dfrac{1}{2} + \dfrac{4}{3} + \dfrac{9}{4} + \cdots$

PROBLEMS: Find the nth term of the series

1. $\dfrac{1}{6} + \dfrac{2}{7} + \dfrac{3}{8} + \cdots$

2. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \cdots$

3. $1 + \dfrac{3}{4} + \dfrac{5}{9} + \cdots$

ANSWERS:

1. $\dfrac{n}{n+5}$

2. $\dfrac{1}{n^3}$

3. $\dfrac{2n-1}{n^2}$

CONVERGENCE

If we find the sum of the first n terms of an infinite series approaches a finite value as $n \to \infty$, then we say the series is convergent. If a series is not convergent, then we say it is divergent.

EXAMPLE: Is the infinite series

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots \text{ convergent?}$$

SOLUTION: Write

$$\lim_{n \to \infty} S_n = S = \frac{a}{1-r}$$

and

$$r = \frac{1}{3}$$

$$a = 1$$

then

$$S = \frac{1}{1 - \frac{1}{3}}$$

$$= \frac{1}{\frac{2}{3}}$$

$$= \frac{3}{2}$$

The limit of the sum of n terms as $n \to \infty$ approaches $\dfrac{3}{2}$, therefore the series is convergent.

EXAMPLE: Is the infinite series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ convergent?}$$

SOLUTION: Write

$$S = \frac{a}{1-r}$$

because this is an infinite geometric series where

$$a = \frac{1}{2}$$

$$r = \frac{1}{2}$$

then

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$= 1$$

The sum has a limit, therefore the series is convergent on 1.

EXAMPLE: Is the infinite series

$$3 + 6 + 12 + \cdots \text{ convergent?}$$

SOLUTION: Find that

$$a = 3$$

and

$$r = \frac{6}{3} = 2$$

Now,

$$|r| > 1$$

therefore

$$\lim_{n \to \infty} S_n = \frac{a - ar^n}{1-r}$$

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and the sum of the series goes to ∞ and the series is divergent.

EXAMPLE: Is the series

$$1 + 3 + 5 + \cdots \text{ convergent?}$$

SOLUTION: Find that the series is arithmetic and

$$a = 1$$

and the nth term is

$$(2n - 1)$$

Consider the nth term as the last term and write

$$S_n = \frac{n}{2}(a + \ell)$$

$$= \frac{n}{2}[1 + (2n - 1)]$$

$$= n^2$$

Then

$$\lim_{n \to \infty} S_n = \infty.$$ Therefore the series is divergent.

PROBLEMS: Determine whether the following series are convergent or divergent.

1. $3 + 6 + 9 + \cdots + 3n + \cdots$

2. $1 + 3 + 9 + \cdots + 3^{n-1} + \cdots$

3. $1 + \dfrac{1}{3} + \dfrac{1}{9} + \cdots + \dfrac{1}{3^{n-1}} + \cdots$

ANSWERS:

1. Divergent

2. Divergent

3. Convergent

If a series is convergent, its nth term must have zero as its limit. But, if the nth term of a series has a limit of zero as $n \to \infty$, this does not mean the series is convergent. If the nth term of a series does not have zero as a limit, then the series is divergent. That is, if the

limit of the nth term is zero, as $n \to \infty$, the series may or may not be convergent.

EXAMPLE: Determine if the series

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots + \cdots + \frac{1}{3 \cdot 2^{n-1}} \text{ is convergent}$$

SOLUTION: Examine the last term and find that

$$\lim_{n \to \infty} \frac{1}{3 \cdot 2^{n-1}} = 0$$

This is a necessary condition, but we must investigate the series further. We find that the series is geometric with

$$|r| < 1$$

Therefore, we conclude the series converges.

EXAMPLE: Determine if the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \text{ is convergent.}$$

NOTE: This is a harmonic series. A harmonic series is a series whose reciprocals form an arithmetic series.

SOLUTION: Investigation of the nth term indicates that

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

We now expand the series by writing

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{n} + \cdots$$

If we group terms as follows:

$$1$$

$$+ \frac{1}{2}$$

$$+ \frac{1}{3} + \frac{1}{4}$$

$$+ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$+ \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}$$

we find

$$1 > \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840} > \frac{1}{2}$$

$$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{2}$$

and if we continue to group terms after the second term in groups of 2, 4, 8, 16, 32, ... we find the sum of each group is greater than $\frac{1}{2}$.
There is an unlimited number of groups, therefore, the limit of the sum

$$\lim_{n \to \infty} S_n = \infty$$

and the series is divergent.

The two previous examples indicate how convergence or divergence of a series is determined. We will use four types of series as reference; that is,

$$a + ar + ar^2 + \cdots + ar^{(n-1)} + \cdots \qquad (1)$$

is convergent, $|r| < 1$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

or

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots \qquad (2)$$

is convergent

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \qquad (3)$$

is divergent, and

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \qquad (4)$$

is convergent, $p > 1$; and is divergent, $p \leq 1$

TEST FOR CONVERGENCE BY COMPARISON

If we know that the series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is convergent and we wish to know if the series

$$B_1 + B_2 + B_3 + \cdots + B_n + \cdots$$

is convergent, we compare the two series term by term. If we find that every term a_i is greater than or equal to every term B_i, that is,

$$a_i \geq B_i$$

then the series under investigation is convergent.

This is because the limit of the sum of terms of the reference series is greater than the limit of the sum of the series under investigation.

EXAMPLE: Test for convergence the series

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots + \frac{1}{5^n} + \cdots$$

SOLUTION: We use the reference series (2) and write

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots \qquad (2)$$

and

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots + \frac{1}{5^n} + \cdots$$

In term by term comparison we find

$$\frac{1}{2} > \frac{1}{5}$$

$$\frac{1}{2^2} > \frac{1}{25}$$

$$\frac{1}{2^3} > \frac{1}{125}$$

$$\frac{1}{2^n} > \frac{1}{5^n}$$

Since the reference series is convergent, the series under investigation is convergent.

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If we desire to test a series for divergence by comparison, we use a reference series which is divergent. Then, if each term of the series under investigation is greater than or equal to the reference series, it too is divergent.

EXAMPLE: Test for divergence the series

$$1 + 3 + 5 + \cdots + (2n - 1) + \cdots$$

SOLUTION: Use the reference series (3); that is,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

and compare, term by term, which yields

$$1 \geq 1$$

$$3 \geq \frac{1}{2}$$

$$5 \geq \frac{1}{3}$$

and

$$2n - 1 \geq \frac{1}{n}$$

then the reference series is divergent, and the series under investigation, term by term, is equal to or greater than the reference and is therefore divergent.

RATIO TEST FOR CONVERGENCE

The ratio test is limited to series where all terms are positive. In this test we must write the nth term and the $(n + 1)^{th}$ term and find the limit of the ratio of these two terms. That is,

$$\lim_{n \to \infty} \frac{t_{(n+1)}}{t_n} = r$$

where $t_{(n+1)}$ is the $(n + 1)^{th}$ term.

If $|r| < 1$ the series is convergent and if $|r| > 1$ the series is divergent. If $|r| = 1$, the test fails because the series could be either convergent or divergent.

EXAMPLE: Test for convergence the series

$$10 + \frac{10^2}{1 \cdot 2} + \frac{10^3}{1 \cdot 2 \cdot 3} + \cdots + \frac{10^n}{1 \cdot 2 \cdot 3 \cdots n} + \cdots$$

SOLUTION: Write the term t_n as

$$\frac{10^n}{1 \cdot 2 \cdot 3 \cdots n}$$

and the term $t_{(n+1)}$ as

$$\frac{10^{n+1}}{1 \cdot 2 \cdot 3 \cdots (n + 1)}$$

The ratio is

$$\frac{\dfrac{10^{n+1}}{1 \cdot 2 \cdot 3 \cdots (n + 1)}}{\dfrac{10^n}{1 \cdot 2 \cdot 3 \cdots n}}$$

Then,

$$\frac{10^{(n+1)}}{1 \cdot 2 \cdot 3 \cdots (n + 1)} \cdot \frac{1 \cdot 2 \cdot 3 \cdots n}{10^n}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots (n + 1)} \cdot \frac{10^{(n+1)}}{10^n}$$

$$= \frac{n!}{(n + 1)!} \cdot \frac{10^{(n+1)}}{10^n}$$

$$= \frac{n!}{(n + 1)!} \cdot \frac{10^{(n+1)-n}}{1}$$

$$= \frac{n!}{(n + 1)!} \cdot \frac{10}{1}$$

$$= \frac{n!}{n!(n + 1)} \cdot \frac{10}{1}$$

$$= \frac{10}{(n + 1)}$$

and

$$\lim_{n \to \infty} \frac{t_{(n+1)}}{t_n}$$

$$= \lim_{n \to \infty} \frac{10}{n + 1}$$

$$= 0$$

Therefore, the $|r| < 1$ and the series is convergent.

EXAMPLE: Test for convergence the series

$$\frac{2}{1^3} + \frac{2^2}{2^3} + \frac{2^3}{3^3} + \cdots + \frac{2^n}{n^3} + \cdots$$

SOLUTION: The t_n term is

$$\frac{2^n}{n^3}$$

and the $t_{(n+1)}$ term is

$$\frac{2^{n+1}}{(n + 1)^3}$$

therefore,

$$\frac{\dfrac{2^{n+1}}{(n + 1)^3}}{\dfrac{2^n}{n^3}}$$

$$= \frac{2^{n+1}}{(n + 1)^3} \cdot \frac{n^3}{2^n}$$

$$= \frac{2^{n+1}}{2^n} \cdot \frac{n^3}{(n + 1)^3}$$

$$= \frac{2}{1} \cdot \frac{n^3}{(n + 1)^3}$$

$$= \frac{2n^3}{(n + 1)^3}$$

$$= 2 \left(\frac{n}{n + 1}\right)^3$$

and

$$2 \left(\frac{n}{n + 1}\right)^3$$

$$= 2 \left[\frac{n}{n + 1} \cdot \left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)\right]^3$$

$$= 2 \left(\frac{1}{1 + \frac{1}{n}}\right)^3$$

Then,

$$\lim_{n \to \infty} 2 \left(\frac{n}{n + 1}\right)^3$$

$$= 2 \lim_{n \to \infty} \left(\frac{n}{n + 1}\right)^3$$

$$= 2 \lim_{n \to \infty} \left(\frac{1}{1 + \frac{1}{n}}\right)^3$$

$$= 2 \,(1)^3$$

$$= 2$$

$$= r$$

and

$$|r| > 1$$

therefore, the series diverges.

PROBLEMS: Test for convergence by the comparison test:

1. $\dfrac{1}{1^2 + 2} + \dfrac{1}{2^2 + 2} + \dfrac{1}{3^2 + 2} + \cdots + \dfrac{1}{n^2 + 2} + \cdots$

Test for convergence by the ratio test:

2. $\dfrac{1}{1(3)} + \dfrac{1}{2(3)^2} + \dfrac{1}{3(3)^3} + \cdots + \dfrac{1}{n(3)^n} + \cdots$

3. $\dfrac{4}{1^3} + \dfrac{4^2}{2^3} + \dfrac{4^3}{3^3} + \cdots + \dfrac{4^n}{n^3} + \cdots$

ANSWERS:

1. Convergent

2. Convergent

3. Diverges